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Chebyshev Approximation with Restricted Ranges by Families with the Betweenness Property

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Let $u < v$ be two given functions. The Chebyshev approximation problem in which all approximants G must satisfy the constraint $u \leqslant G \leqslant v$ is considered. It is assumed that the family $\mathscr{G}$ of (unconstrained) approximants has the betweenness property, satisfied by linear and admissible rational families. A necessary and a sufficient condition for an approximant to be best are obtained. A set on which best approximants agree is obtained. A uniqueness result is given.

Let X be a compact topological space and $C(X)$ the space of real continuous functions on X. For $g \in C(X)$, define

$$\| g \| = \sup\{| g(x)|: x \in X\}.$$

Let u, v be continuous functions from X into the extended real line R with $u < v$. Let $\mathscr{G}$ be a subset of $C(X)$ with elements $F, G, H, \ldots$. Let f be a given element of $C(X)$ and define $E(G, x) = f(x) - G(x)$. The Chebyshev problem with restricted ranges is to choose G^* of $\mathscr{G}$ to minimize $e(G) = \| E(G, \cdot)\|$ subject to the constraint

$$u \leqslant G \leqslant v. \tag{1}$$

Such an element G^* is called a best approximation in $\mathscr{G}$ to f. It will be assumed throughout the discussion that f is fixed, and mention of f is suppressed in the notations $e(G)$ and $E(G, \cdot)$.

We will consider the case in which $\mathscr{G}$ has the betweenness property, which was introduced in [1].

DEFINITION. A subset $\mathscr{G}$ of $C(X)$ has the *betweenness property* if for any two elements G_0 and G_1, there exists a λ-set $\{H_\lambda\}$ of elements of $\mathscr{G}$ such that $H_0 = G_0$, $H_1 = G_1$, and for all $x \in X$, $H_\lambda(x)$ is either a strictly monotonic function of λ or a constant, $0 \leqslant \lambda \leqslant 1$.

Families of functions with the betweenness property include linear families, admissible rational families, and suitable transformations of such families [1, p. 152].

For generality in characterization we will assume that u is lower semicontinuous and v is upper semicontinuous into the extended real line R (for definitions see [3, p. 74–77]).

Let $\mathscr{G}'$ be the set of elements of $\mathscr{G}$ which satisfy the constraint (1). It is obvious from the definition that $\mathscr{G}'$ has the betweenness property if $\mathscr{G}$ does.

LEMMA 1. *Let $\mathscr{G}$ have the betweenness property. Let u be lower semicontinuous and v be upper semicontinuous into the extended real line. Let G_0 be in $\mathscr{G}'$. Let $G_1 \in \mathscr{G}$ and*

$$G_1(x) > u(x) \qquad G_0(x) = u(x)$$
$$G_1(x) < v(x) \qquad G_0(x) = v(x).$$

For all λ-sufficiently small, elements of a λ-set for G_0, G_1 are in $\mathscr{G}'$.

Proof. Let $\{H_\lambda\}$ be a λ-set. Suppose the theorem is false. Then there is a sequence $\{\lambda(k)\} \to 0$ and $\{x_k\}$ such that either

$$H_{\lambda(k)}(x_k) < u(x_k) \tag{2}$$

or

$$H_{\lambda(k)}(x_k) > v(x_k). \tag{3}$$

By taking a subsequence, if necessary, we can ensure that one of (2, 3) always occurs and $\{x_k\}$ has a limit x. Suppose (2) always occurs. We have two possibilities. First we could have $G_0(x) = u(x)$. In this case $G_1(x) > G_0(x)$ and so there is a neighborhood N of x such that

$$G_1(y) > G_0(y) \geqslant u(y) \qquad \text{for} \quad y \in N.$$

This implies that $H_\lambda(y) \geqslant u(y)$ for all $\lambda \in [0, 1]$ and all $y \in N$. This contradicts (2). The second possibility is that $G_0 > u(x)$. There exists $\epsilon > 0$ such that $G_0(x) - u(x) > \epsilon$. By upper semicontinuity of $G - u$ there is a neighborhood N of x such that $G_0(y) - u(y) > \epsilon$ for $y \in N$. As $\{H_{1/k}\}$ converges uniformly to G_0 [1, bottom p. 151], we have $H_\lambda(y) > u(y)$ for $y \in N$ and all λ sufficiently small. This contradicts (2). The case when (3) always occurs is handled by similar arguments.

CHARACTERIZATION OF BEST APPROXIMATION

The set of points at which $E(G, \cdot)$ attains its norm $e(G)$ will be denoted by $M(G)$. By compactness of X and continuity of $| E(G, \cdot)|$, $M(G)$ is nonempty and closed.

THEOREM 1. *A sufficient condition for $G \in \mathscr{G}'$ to be a best approximation is that there exist no element $F \in \mathscr{G}$ such that*

$$\begin{aligned} [f(x) - G(x)][F(x) - G(x)] &> 0 & x &\in M(G) \\ F(x) &\geqslant G(x) & G(x) &= u(x) \\ F(x) &\leqslant G(x) & G(x) &= v(x). \end{aligned} \tag{4}$$

The theorem is obvious.

THEOREM 2. *Let $\mathscr{G}$ have the betweenness property. Let u be lower semicontinuous and v be upper semicontinuous into the extended real line. A necessary condition for $G \in \mathscr{G}'$ to be a best approximation is that there exist no element $F \in \mathscr{G}$ such that*

$$\begin{aligned} [f(x) - G(x)][F(x) - G(x)] &> 0 & x &\in M(G) \\ F(x) &> G(x) & G(x) &= u(x) \\ F(x) &< G(x) & G(x) &= v(x). \end{aligned} \tag{5}$$

Proof. By Lemma 1, if (5) holds there is in the λ-set for (G, F) an element H_λ such that

$$[f(x) - G(x)][H_\lambda(x) - G(x)] > 0 \qquad x \in M(G)$$

and by corollary to Theorem 1 of [1, p. 153], G cannot be best in $\mathscr{G}'$, which has the betweenness property.

The following example shows that the sufficient condition is not necessary.

EXAMPLE. Let $X = [-1, 1]$ and $\mathscr{G}$ be the family of functions of the form αx, α real. Let $u(x) = -x^2$ and $v = +\infty$. Let $f(x) = 1 + x$. The only approximant which is $\geqslant u$ is the zero approximant, which is therefore best. Now $M(0) = \{1\}$ and setting $F(x) = x$, $G(x) = 0$, we have (4) holding. It should be noted that in the case $\mathscr{G}$ is an alternating family on an interval, there is a necessary and sufficient condition for an approximation to be best [2].

A SET ON WHICH BEST APPROXIMATIONS AGREE

From now on we will again assume that u, v are continuous into R. Define

$$\hat{M}(G) = M(G) \vee \{x: G(x) = u(x)\} \vee \{x: G(x) = v(x)\}$$

Continuity ensures that $\hat{M}(G)$ is closed.

Let $\mathscr{G}^*$ be the set of best approximations to f and $N = \bigcap \hat{M}(G)$, $G \in \mathscr{G}^*$. We will show in this section that if $\mathscr{G}^*$ is nonempty then N is nonempty, best approximations must agree on N, but N may not be an error-determining set.

LEMMA 2. *Let $\mathscr{G}$ have the betweenness property and $\mathscr{G}^*$ be nonempty. Given a finite number $G_1, \ldots, G_n$ of elements of $\mathscr{G}^*$ there exists an element G_0 of $\mathscr{G}^*$ such that $\bigcap_{k=1}^{n} \hat{M}(G_k) \supset \hat{M}(G_0)$.*

COROLLARY. *Let $G_0, G_1 \in \mathscr{G}^*$, then the λ-set $\{H_\lambda\}$ for G_0 and G_1 is contained in $\mathscr{G}^*$.*

The proof of these is similar to the proof of the corresponding results in [1, p. 153].

LEMMA 3. *Let $\mathscr{G}$ have the betweenness property. If $\mathscr{G}^*$ is nonempty, N is nonempty.*

The proof is identical to the proof of the corresponding result in [1, p. 154].

LEMMA 4. *Let $\mathscr{G}$ have the betweenness property. Let $G_0, G_1 \in \mathscr{G}^*$, then $G_0(x) = G_1(x)$ for all $x \in N$.*

Proof. Let $G_0, G_1 \in \mathscr{G}^*$ be given and select a λ-set $\{H_\lambda\}$ corresponding to G_0 and G_1, $0 < \lambda < 1$. If $G_0(x) \neq G_1(x)$ then

$$| E(H_\lambda, x)| < \max | E(G_0, x), E(G_1, x)| \qquad 0 < \lambda < 1$$

and $H_\lambda(x)$ is in $(u(x), v(x))$. Since $\{H_\lambda\} \subset \mathscr{G}^*$, $x \notin N$.

One conjecture corresponding to Lemma 6 of [1, p. 154] is

CONJECTURE. Let $\mathscr{G}$ have the betweenness property. If $\mathscr{G}^*$ is nonempty there exists no approximant G such that

$$| E(G, x)| < \rho(f) = \inf\{e(G) : G \in \mathscr{G}\} \qquad x \in N$$

$$u(x) \leqslant G(x) \leqslant v(x) \qquad x \in N.$$

The Conjecture is *false*, for in the example given after Theorem 2, $N = \{0, 1\}$, and if we set $G(x) = x$,

$$| E(G, x)| < 2 = \rho(f) \qquad x \in N$$

$$u(x) \leqslant G(x) \qquad x \in N.$$

It follows that N is not an error determining set.

The result corresponding to Lemma 6 of [1, p. 154] is actually

LEMMA 5. *Let $\mathscr{G}$ have the betweenness property. If $\mathscr{G}^*$ is nonempty there exists no approximant $G \in \mathscr{G}$ such that*

$$| E(G, x)| < \inf\{e(G): G \in \mathscr{G}\} \qquad u(x) < G(x) < v(x) \qquad x \in N \qquad (6)$$

To prove the Lemma we use the arguments of the proof of Lemma 6 of [1, p. 154] supplemented by Lemma 1.

UNIQUENESS

DEFINITION. $\mathscr{G}$ has the sign changing property of degree n at G if for any n distinct points $\{x_1, \dots, x_n\}$ and n real numbers $w_1, \dots, w_n$ which are either $+1$ or -1, there exists an approximant F such that

$$\operatorname{sgn}(F(x_k) - G(x_k)) = w_k \qquad k = 1, \dots, n.$$

We need not specify the closeness of F to G in the above definition since if such an F exists, there exists with the betweenness property such an F arbitrarily close to G.

DEFINITION. $\mathscr{G}$ has property Z of degree n at G if $G - F$ having n zeros implies $F = G$.

Let $\mathscr{G}$ have the betweenness property. The F in the definition of the sign changing property can be chosen such that for given $\epsilon > 0$, $\| F - G \| < \epsilon$. Let $G^* \in \mathscr{G}^*$. If $\mathscr{G}$ has the sign changing property of degree n at G^* then G^* either coincides with the function f being approximated or N has at least $n + 1$ points, for it had less we could find F such that (6) holds, which contradicts Lemma 5. If $\mathscr{G}$ has property Z of degree n at G^* then by Lemma 4 best approximations must be identical if N has n or greater points. We therefore have:

THEOREM 3. *Let $\mathscr{G}$ have the betweenness property and let $G^* \in \mathscr{G}^*$. If $\mathscr{G}$ has property Z of degree $n + 1$ at G^* and the sign changing property of degree n at G^*, then G^* is a unique best approximation.*

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